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# An Inverse Problem for Stochastic Differential Equations

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We discuss the problem of reconstructing the drift coefficient of a diffusion from the knowledge of the transition probabilities outside a given bounded region in $\mathbb{R}^d$, $d > 1$. We also give an interpretation of the solution of this inverse problem in the framework of stochastic mechanics.

**KEY WORDS:** Inverse problem; diffusions; stochastic differential equations; stochastic mechanics; quantum mechanics.

## 1. INTRODUCTION

The study of stochastic differential equations with given coefficients is a well-developed subject from the point of view of theory (see, e.g., refs. 6, 7, 14, 20 and 25) as well as of applications (see, e.g., refs. 1, 8, 17, and 18). In this theory, the "direct" problem of determining properties of the solution in terms of the given coefficients is handled. The corresponding "inverse" problem of determining the coefficients from the knowledge of the process in a given region of state space has not been considered, to our knowledge. For some other kinds of inverse problems, such as the determination of parameters in the sense of statistics, see, however, e.g., refs. 5, 10, 11, 12, and 16. Inverse problems of the former type are on the other hand a well-studied subject in the theory of ordinary differential equations (without

This paper is dedicated to the dear memory of Paola Calderoni.

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stochastic terms) (see, e.g., refs. 19 and 22). It is therefore natural to try to extend this study to the case of stochastic systems. Often in fact it is only possible to observe a process outside some given region and it is natural to ask whether this knowledge is enough to determine all local characteristics of the process. Incidentally, we remark that this problem is quite different from, e.g., extrapolation, filtering, or prediction problems (see, e.g., refs. 3, 11, and 26), which involve knowledge of the process for a given interval of time. There is also another context in which the above inverse problem arises. In the past decade the Euclidean reformulation of quantum theory as a stochastic process has become an indispensable tool for the solution of important problems ranging from mathematical physics to elementary particle phenomenology. These successes of Euclidean quantum theory have inevitably led to questions about its fundamental nature—is there something physical to the stuff that diffuses according to the laws of Euclidean quantum theory? In particular: is there a scattering theory for this stuff, and what does it have to do with quantum scattering theory? There are at this time partial answers.[9,17,23] Here we focus on the stochastic mechanical ground-state process (in the sense of, e.g., refs. 4, 17, and 24). We can reconstruct the quantum theory, e.g., from its invariant density,[8,17] but its asymptotic behavior, in regions far away from the scattering center, tells us little—at best the scattering length—about the scattering of the quantum particles.[2] A related question which is connected with the general inverse problem mentioned above is the following. Assume that we observe the sample paths of the ground-state process only outside a bounded domain $\Lambda$: can we reconstruct the process inside $\Lambda$? We can think of observing Brownian motion under a microscope where a black spot obscures a part $\Lambda$ of the field of vision. We observe the sample paths as they vanish and reappear from under the black spot and would like to learn from these observations how the Brownian particles behave in the hidden region, i.e., what obstacles (forces, drifts) they encounter there. More precisely, this problem, which is the one we mentioned at the beginning, can be formulated as follows.

Let us suppose that we know the process: $X_t$ is a solution of the stochastic differential equation $dX_t = \beta(X_t, t)\,dt + dW_t$ for all $t$ such that $X_t(\cdot) \in \Lambda^c \equiv \mathbb{R}^d \backslash \Lambda$. Our goal is to reconstruct the drift field $\beta$ which is responsible for the diffusion. In the case of stochastic mechanics, it is a gradient field, i.e., $\beta = \nabla\varphi$, where $\varphi$ can be deduced easily from the solution of the Schrödinger equation. In the stationary case the corresponding potential $V$ is given by $2V = \Delta\varphi + |\nabla\varphi|^2$. Hence in this case the problem is as follows: Let $\Lambda$ be any bounded open domain in $\mathbb{R}^d$ and let $V$: $\mathbb{R}^d \to \mathbb{R}$ be a potential such that supp $V \cap \Lambda \neq 0$. Is it possible to reconstruct $V$ acting in $\Lambda$ from observations on the solution process $X$, made only in $\mathbb{R}^d \backslash \Lambda$?

In this paper we give a solution of this inverse problem in the case $d>1$. In Section 2 we give a representation formula for the transition probabilities of the process. In Section 3 we use this representation for determining the local characteristics of the process.

## 2. A REPRESENTATION FORMULA FOR THE TRANSITION PROBABILITIES

Let $X_t$ be a solution of the stochastic differential equation (in the Ito sense)

$$dX_t = \beta(X_t)\,dt + dW_t, \qquad X_t = x \tag{2.1}$$

with $W_t$ a Brownian motion $R^d$ starting at 0. We assume that $\beta$ is continuously differentiable. Then the SDE (2.1) has a pathwise unique solution. We assume, moreover, that the solution to (2.1) has no explosion.

Let $\psi \in C_0^1(R^d)$ such that $\psi(z)=1$ for $z \in R^d$ with $|z| \leqslant 1$, and let $\psi_n(z) = \psi(n^{-1}z)$. Also let us consider the following SDE:

$$dX_t^n = \psi_n(X_t^n)\,\beta(X_t^n)\,dt + dW_t, \qquad X_t^n = x \tag{2.2}$$

Then this SDE (2.2) also has a pathwise unique solution. Therefore by the Cameron–Martin–Maruyama–Girsanov formula we have

$$\begin{aligned}
&E[f(X_t), \|X\|_t \leqslant n] \\
&\quad = E[f(X_t^n), \|X^n\|_t \leqslant n] \\
&\quad = E_x\left\{f(W_t)\exp\left[\int_0^t \psi_n(W_s)\,\beta(W_s)\,dW_s \right.\right. \\
&\qquad \left.\left. -\frac{1}{2}\int_0^t |\psi_n(W_s)\,\beta(W_s)|^2\,ds\right], \|W\|_t \leqslant n\right\} \\
&\quad = E_x\left\{f(W_t)\exp\left[\int_0^t \beta(W_s)\,dW_s - \frac{1}{2}\int_0^t |\beta(W_s)|^2\,ds\right], \|W\|_t \leqslant n\right\}
\end{aligned}$$

for $f \in C_0(R^d)$, where the expectation $E_x$ is with respect to Brownian motion starting from $x$ at time 0, and we denote $\max_{0 \leqslant s \leqslant t} |g(s)|$ by $\|g\|_t$.

So we have

$$E_x\left\{\exp\left[\int_0^t \beta(W_s)\,dW_s - \frac{1}{2}\int_0^t |\beta(W_s)|^2\,ds\right]\right\} = 1, \qquad t \geqslant 0$$

and

$$E[f(X_t)] = E_x\left\{f(W_t)\exp\left[\int_0^t \beta(W_s)\,dW_s - \frac{1}{2}\int_0^t |\beta(W_s)|^2\,ds\right]\right\}$$

for any $f \in C_0(R^d)$.

Assume now that $\beta$ is a gradient field, i.e., $\beta(\cdot) = \nabla\varphi(\cdot)$ for some $C^2$ function $\varphi$ and that $2V = \Delta\varphi + |\nabla\varphi|^2$ is lower bounded. Then, by using Ito's formula, we have

$$E[f(X_t)] = E_x\left[f(W_t)\exp\left\{[\varphi(W_t) - \varphi(W_0)] - \int_0^t V(W_s)\,ds\right\}\right]$$

So, setting

$$p_t(x, y) \equiv \{\exp[\varphi(y) - \varphi(x)]\}\left(\frac{1}{2\pi t}\right)^{d/2}\exp\left(-\frac{1}{2t}|x-y|^2\right)$$
$$\times E_{0,x}^{t,y}\left\{\exp\left[-\int_0^t V(W_s)\,ds\right]\right\} \tag{2.3}$$

for each $t > 0$, $x, y \in R^d$ (where the expectation is with respect to a Brownian bridge starting at time 0 from $x$ and ending at time $t$ in $y$), it is easy to see that $p_t(x, y)$ is continuous in $x$ and $y$, and

$$E[f(X_t)] = \int p_t(x, y) f(y)\,dy. \tag{2.4}$$

Let $\gamma_{xy}(s)$, $s \in [0, t]$ be the straight line from $x$ to $y$, i.e.,

$$\gamma_{xy}(s) = \frac{s}{t}(y - x) + x \tag{2.5}$$

Then we have

$$\varlimsup_{t \downarrow 0} t \cdot \log P_{0,x}^{t,y}[\|W - \gamma_{x,y}\|_t > \varepsilon] < 0$$

for any $\varepsilon > 0$.

Also, we have, if $\|W - \gamma_{x,y}\|_t \leqslant \varepsilon$, that

$$\left|\int_0^t V(W_s)\,ds - \int_0^t V(\gamma_{x,y})\,ds\right|$$
$$\leqslant t \cdot \max\{|V(\gamma_{x,y}(s)) - V(\gamma_{x,y}(s) + z)|;\, s \in [0, t],\, z \in R^d,\, |z| \leqslant \varepsilon\}$$

Therefore,

$$\overline{\lim}_{t\downarrow 0}\left|\frac{1}{t}\cdot\log E_{0,x}^{t,y}\left\{\exp\left[-\int_0^t V(W_s)\,ds\right]\right\}-\int_0^1 V(s(y-x)+x)\,ds\right|$$

$$=\overline{\lim}_{t\downarrow 0}\left|\frac{1}{t}\cdot\log E_{0,x}^{t,y}\left[\left\{\exp\left[-\int_0^t V(W_s)\,ds\right]+\int_0^t V(\gamma_{x,y}(s))\,ds\right\}\right]\right|$$

$$\leqslant \max\Bigg(\Big(-\inf V+\max_{0\leqslant s\leqslant t} V(\gamma_{x,y}(s))\Big)$$

$$\times\overline{\lim}_{t\downarrow 0}\frac{1}{t}\log P_{0,x}^{1,y}[\|W-\gamma_{x,y}\|_t>\varepsilon]\;\overline{\lim}_{t\downarrow 0}\frac{1}{t}\log E_{0,x}^{t,y}$$

$$\times\left\{\exp\left[\left|\int_0^t V(W_s)\,ds-\int_0^t V(\gamma_{x,y})\,ds\right|\right],\ \|W-\gamma_{x,y}\|_t\leqslant\varepsilon\right\}\Bigg)$$

$$\leqslant \max\{|V(s(y-x)+x)-V(s(y-x)+x+z)|;$$

$$s\in[0,1],\ z\in R^d,\ |z|\leqslant\varepsilon\}$$

for any $\varepsilon>0$. Since the last term converges to zero as $\varepsilon\downarrow 0$, we see that

$$\log E_{0,x}^{t,y}\left\{\exp\left[-\int_0^t V(W_s)\,ds\right]\right\}=t\int_0^1 V(s(y-x)+x)\,ds+o(t)$$

Thus we have the following.

**Theorem 2.1.** Let $\varphi\in C^2$ and let $2V=\Delta\varphi+|\nabla\varphi|^2$ be lower bounded. Then the transition probability density $p_t(x,y)$ of the solution to

$$dX_t=\nabla\varphi(X_t)\,dt+dW_t$$

is continuous and given by

$$p_t(x,y)=\{\exp[\varphi(y)-\varphi(x)]\}\left(\frac{1}{2\pi t}\right)^{d/2}\exp\left(-\frac{1}{2t}|x-y|^2\right)$$

$$\times E_{0,x}^{t,y}\left\{\exp\left[-\int_0^t V(W_s)\,ds\right]\right\}$$

the expectation being with respect to the Brownian bridge starting at $t=0$ in $x$ and ending at time $t$ in $y$.

Moreover, for $t\downarrow 0$ we have

$$p_t(x,y)=\left(\frac{1}{2\pi t}\right)^{d/2}\exp\left(-\frac{1}{2t}|x-y|^2\right)$$

$$\times\exp\left\{[\varphi(y)-\varphi(x)]-t\int_0^1 V(s(y-x)+x)\,ds+o(t)\right\}$$

In Section 3 we shall use the above estimate in the following way. Let $\Lambda$ be a bounded domain in $R^d$ with smooth boundary. Suppose we know $p_t(x, y)$ for all $x, y \in \partial\Lambda$ and all $0 < t < +\infty$. Then we know $\varphi(y) - \varphi(x)\ \forall x, y \in \partial\Lambda$, since

$$\lim_{t \downarrow 0} \frac{p_t(x, y)}{p_t^0(x, y)} = \exp[\varphi(y) - \varphi(x)] \tag{2.6}$$

with

$$p_t^0(x, y) = (2\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{2t}\right)$$

Also, we know $\int_0^1 V(s(y-x)+x)\, ds\ \forall x, y \in \partial\Lambda$, since

$$\lim_{t \downarrow 0} \frac{1}{t}\left\{\log\left(\frac{p_t(x, y)}{p_t^0(x, y)}\right) - [\varphi(y) - \varphi(x)]\right\} = -\int_0^1 V(s(y-x)+x)\, ds$$

## 3. RECONSTRUCTION OF THE DRIFT AND OF THE INTERACTION POTENTIAL

Let $\varphi$ and $V$ be as in Section 2. As we saw at the end of Section 2, the knowledge of $p_s(x, y)$, $\forall x, y \in \partial\Lambda$ and all $0 \leqslant s < t$, yields in particular

$$F(x, y) \equiv \int_0^1 V(x + (y-x)\, s)\, ds, \qquad x, y \in \partial\Lambda$$

We shall remark that this is equivalent to the knowledge of

$$G(a, b) \equiv \int_{\mathbb{R}} \chi_\Lambda(a+sb)\, V(a+sb)\, ds \tag{3.1}$$

for all $a, b \in \mathbb{R}^d$, $b \neq 0$. In fact,

$$G(a, b) = \int_{\{s \mid a+sb \in \Lambda\}} V(a+sb)\, ds = F(x, y) \tag{3.2}$$

with $x, y$ the points where the straight line $[a+sb, s \in \mathbb{R}_+]$ intersects $\partial\Lambda$. We shall now show that this also implies the knowledge of $V$ inside $\Lambda$, at least for $d > 1$. In fact, let us consider the Fourier transform of $V_\Lambda \equiv \chi_\Lambda V$, i.e.,

$$\tilde{V}_\Lambda(p) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ipx} V_\Lambda(x)\, dx, \qquad p \in \mathbb{R}^d \tag{3.3}$$

Let $d \geqslant 2$ and set, for $p \neq 0$,

$$H_p = \{x \in \mathbb{R}^d \mid p \cdot x = 0\}$$

We can then write, by the Fubini theorem, using the decomposition

$$\mathbb{R}^d = H_p \oplus \{p\} = H_p \oplus \{t\hat{p},\ t \in \mathbb{R},\ \hat{p} = p/|p|\}$$

and the change of coordinates

$$x \in \mathbb{R}^d \to (z, t\hat{p})$$

with $z \in H_p$, $t\hat{p} \in \{p\}$, $t \in \mathbb{R}$, remarking that the volume measure $dx$ becomes $dy\, dt$,

$$\tilde{V}_\Lambda(p) = (2\pi)^{-d/2} \int_{\mathbb{R}} \left[ \int_{H_p} e^{-i(t\hat{p} \oplus z)p} \chi_\Lambda(t\hat{p} \oplus z)\, V(t\hat{p} \oplus y)\, dy \right] dt \qquad (3.4)$$

Using the fact that $p \cdot y = 0$, we obtain

$$\tilde{V}_\Lambda(p) = (2\pi)^{-d/2} \int_{\mathbb{R}} e^{-it|p|} \left[ \int_{H_p} \chi_\Lambda(t\hat{p} \oplus z)\, V(t\hat{p} \oplus z)\, dz \right] dt \qquad (3.5)$$

But

$$\int_{H_p} \chi_\Lambda(t\hat{p} \oplus z)\, V(t\hat{p} \oplus z)\, dz = \int_{D_\Lambda} V(t\hat{p} \oplus z)\, dz \qquad (3.6)$$

with $D_\Lambda \equiv \{z \in H_p,\ t\hat{p} \oplus z \in \Lambda\}$. But

$$\begin{aligned} \int_{D_\Lambda} V(t\hat{p} \oplus z)\, dz &= \int_{D_\Lambda} V(t\hat{p} \oplus z)\, dz_1 \cdots dz_{d-1} \\ &= \int_{D_\Lambda} V(t\hat{p} \oplus s_1 \hat{z}_1 \oplus \cdots \oplus s_{d-1} \hat{z}_{d-1})\, ds_1 \cdots ds_{d-1} \qquad (3.7) \end{aligned}$$

with $\hat{z} \equiv z/|z|$.

Using the above observation with respect to the $s_{d-1}$ integration and then performing successively the $s_{d-2}, \ldots, s_1$ integrations, we see that (3.7) is a known function $F(t, \hat{p})$ [determined by the knowledge of $p_s(x, y)$, $\forall x, y \in \partial\Lambda, 0 \leqslant s \leqslant t$]. Hence, inserting this into (3.5), we get that

$$\tilde{V}_\Lambda(p) = (2\pi)^{-d/2} \int_{\mathbb{R}} e^{-it|p|} F(t, \hat{p})\, dt$$

is a known function of $p$, for all fixed $p \in \mathbb{R}$, $p \neq 0$.

By the inverse Fourier transformation we then get $V_\Lambda(x)$ for all $x \in \mathbb{R}^d$, as determined by the above knowledge of $p_t(x, y)$. We formulate these results in the following proposition.

**Proposition 3.1.** Let $\Lambda$ be a bounded domain of $\mathbb{R}^d$, $d > 1$, with smooth boundary $\partial\Lambda$. Let $p_t(x, y)$ be the transition probability density for a diffusion process with unit diffusion coefficient and drift given by $\beta = \nabla\varphi$, satisfying the regularity conditions as in Section 2. Then from the knowledge of $p_s(x, y)$, $\forall x, y \in \partial\Lambda$, $0 \leqslant s \leqslant t$, for some $t > 0$, we get uniquely $\varphi(y) - \varphi(x)$, $\forall x, y \in \partial\Lambda$ [by (2.6)] and $2V(x) \equiv \Delta\varphi(x) + |\nabla\varphi(x)|^2$ for all $x \in \Lambda$, by the Fourier transform method given above.

We finally remark that this proposition permits us to determine the drift $\beta(x)$ for all $x \in \Lambda$. In fact, let us fix $x_0 \in \partial\Lambda$. Define $\Phi(x) \equiv \exp[\varphi(x) - \varphi(x_0)]$, $\forall x \in \partial\Lambda$. Then we have

$$\Delta\Phi(x) = [\Delta\varphi(x) + |\nabla\varphi(x)|^2]\, \Phi(x) = 2V(x)\, \Phi(x)$$

for all $x \in \Lambda$ (by the definition of $V$).

By Proposition 3.1, $p_s(x, y)$, $0 \leqslant s \leqslant t$, $x, y \in \partial\Lambda$, determines $\Phi(x)$ for $x \in \partial\Lambda$ and $V(x)$ for $x \in \Lambda$.

The boundary value problem

$$\begin{aligned} \Delta\Phi(x) &= 2V(x)\, \Phi(x), && x \in \Lambda \\ \Phi(x) &= e^{\varphi(x) - \varphi(x_0)}, && x \in \partial\Lambda \end{aligned}$$

has, however, a unique solution, under our assumptions, which yields $\Phi(x)$ for all $x \in \Lambda$. Hence we have the following result.

**Theorem 3.2.** Let $\Lambda$ be a bounded domain of $\mathbb{R}^d$, $d > 1$, with smooth boundary. Let $p_t(x, y)$ be the transition probability density of a diffusion process with unit diffusion coefficient and drift being a gradient field $\beta = \nabla\varphi$ satisfying regularity conditions as in Section 2. Then $\beta(x)$, $x \in \Lambda$, is uniquely determined by the knowledge of $p_t(x, y)$ for $x, y \in \partial\Lambda$.

*Remark 1.* Our method covers only the case $d > 1$. Results for the case $d = 1$ should, however, be obtainable by other methods, using, e.g., ref. 15 or a discrete approximation.

*Remark 2.* In recent years, ideas and methods of the theory of stochastic processes have been used quite extensively in the study of quantum mechanics. In turn, the theory of stochastic processes has received many stimulating impulses from quantum theory. See, e.g., refs. 1, 2, 4, 8, 9, 13, 17, 21, 24 and 27. As mentioned in the introduction, the problem we discuss in this paper has a very natural physical application through the interpretation of quantum mechanics as stochastic mechanics (see, e.g., refs. 8 and 17).

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